THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
Tutorial 10

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1. (Series of Numbers) Consider $\sum_{n=1}^{\infty}\{a_n\}$.

   (a) State the definition and Cauchy criterion of convergence.

   (b) Show that if $\sum_{n=1}^{\infty} a_n$ converges in $\mathbb{R}$, then in particular, $\lim_{n\to\infty} a_n = 0$. Show that the converse does not hold.

   (c) State the rearrangement theorem for a conditionally convergent series.

2. (a) State the comparison test.

   (b) State the ratio test and root test. (In the tutorial the statement was not accurate)

   Theorem 1 (Ratio Test). *Let $\{a_n\}$ be nonzero and suppose the following limit exists:*

   $$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \in [0,\infty].$$

   Then:

   i. *If $0 \le L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.*
   ii. *If $1 < L \le \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.*
   iii. *If $L = 1$, then the test is inconclusive.*

   The statement regarding root test is similar:

   Theorem 2 (Root Test). *Let $\{a_n\}$ be nonzero and suppose the following limit exists:*

   $$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} = L \in [0,\infty].$$

   Then:

   i. *If $0 \le L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.*
   ii. *If $1 < L \le \infty$, then $\sum_{n=1}^{\infty} a_n$ diverges.*
   iii. *If $L = 1$, then the test is inconclusive.*

   (c) State the alternating series test.

   (d) State the integral test.

   (e) Use the definition and convergence (divergence) tests, study the convergence of the following series: if possible, study whether they are absolutely or conditionally convergent.

   i. $\sum_{n=1}^{\infty} \frac{1}{n}$. This is called harmonic series.
   ii. $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p \in \mathbb{R}$. This is called Riemann zeta function (on the positive real axis if $p > 1$).

iii. $\sum_{n=3}^{\infty} \frac{1}{n(\ln(n))^p}$.

iv. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$.

v. $\sum_{n=10}^{\infty} \frac{(-1)^n}{\ln n}$.

vi. $\sum_{n=1}^{\infty} n!e^{-n}$.

vii. $\sum_{n=1}^{\infty} n!e^{-n^2}$.

viii. $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$.

ix. $0 < a < 1$ and $a^2 + a + a^4 + a^3 + \ldots$. This example shows that root test is strictly stronger than ratio test in some sense.

(Not Discussed)

3. (a) Let $\{x_n\}, \{y_n\}$ be given and let $s_n := \sum_{k=1}^{n} y_k$, $s_0 := 0$. Prove the summation by parts formula:

$$\sum_{k=n+1}^{m} x_k y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k,$$

for $m > n$.

(b) Use summation by parts formula to prove the Kronecker's Lemma: Let

$$\sum_{n=1}^{\infty} x_n = s \in \mathbb{R}$$

Let $0 < b_1 \leq b_2 \leq \cdots \leq b_n \to \infty$. Then

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{k=1}^{n} b_k x_k = 0.$$

Note in particular, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k = 0.$$

(c) Use summation by parts formula to study the convergence of the series:

$$\sum_{n=1}^{\infty} \frac{\cos \pi n}{n}$$

Hint: Write $\cos(\pi n) = \mathrm{Re}(e^{i\pi n})$.

4. (Just for fun) By formal algebraic manipulations, show that:

(a) $1 - 1 + 1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$.

(b) $1 - 2 + 3 - 4 + 5 - 6 + \cdots = \frac{1}{4}$.

(c) $1 + 2 + 3 + 4 + 5 + 6 + \cdots = -\frac{1}{12}$.

Warning: They all diverge in our definition! They make sense only if we generalise the definitions. Google for abelian and Tauberian's theorems.